1. **Title**

The mono-anabelian geometry of geometrically pro-p arithmetic fundamental groups of associated low-dimensional configuration spaces I - II - III

Kazumi Higashiyama

- [Hgsh2], to appear in PRIMS (My doctoral thesis)

Part I 2 July, 2021
Part II 8 July, 2021
Part III 8 July, 2021

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2. **Abstract**

Let p be a prime number. In this talk, we study geometrically pro-p arithmetic fundamental groups of low-dimensional configuration spaces associated to a given hyperbolic curve over an arithmetic field such as a number field or a p-adic local field. Our main results concern the group-theoretic reconstruction of the function field of certain tripods (i.e., copies of the projective line minus three points) that lie inside such a configuration space from the associated geometrically pro-p arithmetic fundamental group, equipped with the auxiliary data constituted by the collection of decomposition groups determined by the closed points of the associated compactified configuration space.

3. **Part I**

We explain the statement of Mochizuki's results and today's main results. Finally, we explain "From PGCS to CFS" in the proof of main results. By this explanation, we confirms what kind of operation mono-anabelian geometry.

4. **Part II, Part III**

We explain the idea of Main Theorem in comparison with the past results. After that, we explain Main Theorems as possible as.

- Review of Part I
- Idea (p.23-26)
- From CFS to base fields (p.28-38)
- From trip. very ample (g, r, n) PGCS to $(0, 3, 2)$ PGCS (p.39-45)
- Semi-absolute bi pro-p GC (p.46-47)
- Reconstruct function fields (p.48-52)

5. **What is Grothendieck Conjecture**

What is Grothendieck conjecture (GC): omit

k: a suitable field

${}^{\dagger}U$, ${}^{\ddagger}U$: suitable varieties

Then is the natural map

$$\mathrm{Isom}_k({}^{\dagger}U, {}^{\ddagger}U) \to \mathrm{Isom}_{G_k}(\pi_1({}^{\dagger}U), \pi_1({}^{\ddagger}U))/\mathrm{Inn}$$

bijective?

6. **various GC**

various choice

profinite $\Longleftarrow$ geom. pro-p $\Longleftarrow$ pro-p

bi $\Longleftarrow$ mono

relative $\Longleftarrow$ semi-absolute $\Longleftarrow$ absolute

• Past results
Mochizuki, [Topics], Theorem 4.12: relative bi geom. pro-Σ GC
Mochizuki, [AbsTpIII], Corollary 1.10, (absolute) mono profinite GC

$\vdots$

• Today's main results
Higashiyama, [Hgsh2], Theorem 0.1, semi-absolute bi pro-p GC
Higashiyama, [Hgsh2], Theorem 0.2, (semi-absolute) mono pro-p GC

7. Notation

p: a prime number
n: a positive integer
k: a generalized sub-p-adic filed
($\overset{\text{def}}{\Longleftrightarrow}$ $k \overset{\exists}{\hookrightarrow}$ finitely generated $/(\mathbb{Q}_p^{\text{unr}})^\wedge$)
$\bar{k}$: the algebraic closure
$X^{\log}/k$: an ordered hyperboric log curve of type (g, r) $(2g - 2 + r > 0)$
$T^{\log}/k$: an ordered hyperboric log curve of type $(0, 3)$
$X_n^{\log}$: the n-th log configuration space of type (g, r, n)
X: the underlying scheme of $X^{\log}$ (proper curve)
U_X: the interior of $X^{\log}$ (hyperbolic curve of type (g, r))
$\pi_1(U_X)$: the étale fundamental group
$G_k \overset{\text{def}}{=} \mathrm{Gal}(\bar{k}/k)$

8. **Detail: Exact sequence**

We consider

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & \pi_1(U_{X_n} \times_k \bar{k}) & \longrightarrow & \pi_1(U_{X_n}) & \longrightarrow & G_k & \longrightarrow & 1 & & \text{(profinite)} \\
 & & \downarrow & \subset & \downarrow & & \| & & & & \\
1 & \longrightarrow & \pi_1(U_{X_n} \times_k \bar{k})^{(p)} & \longrightarrow & \pi_1(U_{X_n})^{[p]} & \longrightarrow & G_k & \longrightarrow & 1 & & \text{(geom. pro-}p\text{)} \\
 & & \| & & \downarrow & & \downarrow & & & & \\
 & & \pi_1(U_{X_n} \times_k \bar{k})^{(p)} & \longrightarrow & \pi_1(U_{X_n})^{(p)} & \longrightarrow & G_k^{(p)} & \longrightarrow & 1 & & \text{(pro-}p\text{)}
\end{array}$$

When we consider the pro-p exact sequence, we assume that

$$\pi_1(U_{X_n} \times_k \bar{k})^{(p)} \subseteq \pi_1(U_{X_n})^{(p)}$$

Note that

$$\pi_1(U_{X_n} \times_k \bar{k})^{(p)} \subseteq \pi_1(U_{X_n})^{(p)} \qquad \Longrightarrow \qquad \zeta_p \in k$$

In general,

$$\text{profinite} \qquad \Longleftarrow \qquad \text{geom. pro-}p \qquad \Longleftarrow \qquad \text{pro-}p$$

9. **Detail: bi? mono?-1**

Lemma

If k: a p-adic local field, then by local class field theory, we have the local reciprocity map $\rho_k : k^\times \hookrightarrow G_k^{\text{ab}}$ which induces an isom. $\widehat{k^\times} \xrightarrow{\sim} G_k^{\text{ab}}$

magenta color is an abstruct group-like object

Definition(mono)

$$G_k \rightsquigarrow \widehat{k^\times}$$

means

Let G_k be a profinite group which is isom. to G_k

$\widehat{k^\times} \overset{\text{def}}{=} G_k^{\text{ab}}$ (only magenta object and group-theoretic operation)

Then any isom. $G_k \xrightarrow{\sim} G_k$ induces a commutative diagram

$$\begin{array}{ccc} G_k & \xrightarrow{\sim} & G_k \\ \downarrow & & \downarrow \\ \widehat{k^\times} & \xrightarrow{\sim} & \widehat{k^\times} \end{array}$$

10. **Detail: bi? mono?-2**

In particular, assertion $G_k \rightsquigarrow \widehat{k^\times}$ follows immediately from Lemma and Construction

group	geometry
G_k	G_k
$\widehat{k^\times} \overset{\text{def}}{=} G_k^{\text{ab}}$ ······ coincide ······	$\widehat{k^\times} \overset{\sim}{\to} G_k^{\text{ab}}$
Construction	Lemma

11. **Detail: bi? mono?-3**

Proposition(mono-reconstruction)

$$G_k \rightsquigarrow \widehat{k^\times}$$

(pf) It follows from Lemma and Construction

Proposition(bi-reconstruction)
Any isom. $G_k \xrightarrow{\sim} G_k$ induces a commutative diagram

$$\begin{array}{ccc} G_k & \xrightarrow{\sim} & G_k \\ \downarrow & & \downarrow \\ \widehat{k^\times} & \xrightarrow{\sim} & \widehat{k^\times} \end{array}$$

(pf) It follows from Lemma

In general,

$$\text{bi} \qquad \Longleftarrow \qquad \text{mono}$$

bi GC

$$\mathrm{Isom}_k(U_{{}^\dagger X}, U_{{}^\ddagger X}) \to \mathrm{Isom}_{G_k}(\pi_1(U_{{}^\dagger X}), \pi_1(U_{{}^\ddagger X}))/\mathrm{Inn}$$

is bijective?

mono GC

$$\pi_1(U_X) \rightsquigarrow \ \mathrm{Fnct}(U_X) \text{ or } U_X?$$

12. **Detail: relative? semi-absolute? absolute?**

• relative: Is the morphism

$$\mathrm{Isom}_k(U_X, U_X) \to \mathrm{Isom}_{G_k}(\pi_1(U_X), \pi_1(U_X)/\mathrm{Inn}$$

bijective? i.e.,

$U_X \overset{\sim}{\to} U_X$ vs.
$$\begin{array}{ccc} \pi_1(U_X) & \longrightarrow & \pi_1(U_X) \\ & \searrow \quad \swarrow & \\ & G_k & \end{array}$$

• semi-absolute:

$U_X \overset{\sim}{\to} U_X$ vs.
$$\begin{array}{ccc} \pi_1(U_X) & \longrightarrow & \pi_1(U_X) \\ \downarrow & & \downarrow \\ G_k & \longrightarrow & G_k \end{array}$$

• absolute:

$U_X \overset{\sim}{\to} U_X$ vs. $\pi_1(U_X) \overset{\sim}{\to} \pi_1(U_X)$

In general,

$$\text{relative} \quad \Longleftarrow \quad \text{semi-absolute} \quad \Longleftarrow \quad \text{absolute}$$

13. **Main Theorem and flowchart of reconstruction-1**

trip. very ample (g,r,n) PGCS $(\pi_1(U_{X_n})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_n})$

⋮ Part III

$(0,3,2)$ PGCS $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

⋮ Part I

CFS $(T_2(K), T(K), T(K) \setminus U_T(K), \mathrm{Aut}_k(U_{T_2}), \{T_2(K) \to T(K)\})$

⋮ Part II

base fields $k,\ K$

⋮ Part III

a function field $\mathrm{Fnct}(U_{T_2})$

14. **Main Theorem and flowchart of reconstruction-2**

Theorem 0.1 (Part III) semi-abs. bi pro-p GC

$$\mathrm{Isom}(U_{{}^{\dagger}X_n}, U_{{}^{\ddagger}X_n}) \xrightarrow{\sim} \mathrm{Isom}({}^{\dagger}\mathrm{PGCS}, {}^{\ddagger}\mathrm{PGCS})/\mathrm{Inn}$$

Theorem 0.2 (Part III) mono pro-p GC

trip. very ample PGCS $(\pi_1(U_{X_n})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_n}) \rightsquigarrow \mathrm{Fnct}(U_{T_2})$

Theorem 0.3 (Part I; Part II)

$$(0,3,2)\ \mathrm{PGCS}\ (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2}) \rightsquigarrow \ \mathrm{CFS} \ \rightsquigarrow k,\ K$$

Theorem 0.4 (Part III)

$$(0,3,2)\ \mathrm{P}GCS\ (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2}) \underset{Thm\ 0.3}{\rightsquigarrow} k$$

$$\bigcap\ inclusion$$

$$\pi_1(U_T) \underset{\text{[AbsTpIII], 1.10}}{\rightsquigarrow} k \subseteq \mathrm{Fnct}(U_T)$$

15. **Comparison**

• Past results

Mochizuki, [Topics], Theorem 4.12: relative bi geom. pro-Σ GC

$$\mathrm{Isom}_k(U_{^\dagger X}, U_{^\ddagger X}) \to \mathrm{Isom}_{G_k^\Sigma}(\pi_1(U_{^\dagger X})^\Sigma, \pi_1(U_{^\ddagger X})^\Sigma)/\mathrm{Inn}$$

Mochizuki, [AbsTpIII], Corollary 1.10, (absolute) mono profinite GC

$$\pi_1(U_X) \rightsquigarrow \mathrm{Fnct}(U_X)$$

• Today's main results

Higashiyama, [Hgsh2], Theorem 0.1, semi-absolute bi pro-p GC

$$\mathrm{Isom}(U_{^\dagger X_n}, U_{^\ddagger X_n}) \to \mathrm{Isom}(^\dagger\mathrm{PGCS}, {}^\ddagger\mathrm{PGCS})/\mathrm{Inn}$$

Higashiyama, [Hgsh2], Theorem 0.2, (semi-absolute) mono pro-p GC

trip. very ample PGCS $(\pi_1(U_{X_n})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_n}) \rightsquigarrow \mathrm{Fnct}(U_{T_2})$

16. **From PGCS to CFS-1**

Definition

k: a generalized sub-p-adic field

$k \subseteq K \subseteq \bar{k}$: the maximal pro-$p$ subextension of $\bar{k}/k$

$X^{\log}/k$: an ordered hyperboric log curve of type (g, r) $\quad (2g - 2 + r > 0)$ (cf. Notation)

Suppose that

$$1 \longrightarrow \pi_1(U_{X_n} \times_k \bar{k})^{(p)} \longrightarrow \pi_1(U_{X_n})^{(p)} \longrightarrow G_k^{(p)} \longrightarrow 1$$

$\pi_1(U_{X_n})^{(p)}$: a profinite group

$G_k^{(p)}$: a quotient group of $\pi_1(U_{X_n})^{(p)}$

$\mathcal{D}_{X_n}$: a set of subgroups of $\pi_1(U_{X_n})^{(p)}$

We shall refer to $(\pi_1(U_{X_n})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_n})$ as a (g, r, n) PGCS-collection if there exists a collection of data as follows:

an isomorphism

$$\gamma \colon \pi_1(U_{X_n})^{(p)} \xrightarrow{\sim} \pi_1(U_{X_n})^{(p)}$$

such that γ induces a commutative diagram

$$\begin{array}{ccc} \pi_1(U_{X_n})^{(p)} & \xrightarrow{\sim} & \pi_1(U_{X_n})^{(p)} \\ \downarrow & \circlearrowright & \downarrow \\ G_k^{(p)} & \overset{\sim}{\dashrightarrow} & G_k^{(p)}, \end{array}$$

and a bijection

$$\mathcal{D}_{X_n} \xrightarrow{\sim} \mathcal{D}_{X_n} \overset{\text{def}}{=} \{D \subseteq \pi_1(U_{X_n})^{(p)} \mid D \text{ is a decomposition group associated to some } x \in X_n(K)\}.$$

17. **From PGCS to CFS-2**

START: $(0, 3, 2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

$\pi_1(U_{T_2} \times_k \bar{k})^{(p)} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\pi_1(U_{T_2})^{(p)} \to G_k^{(p)})$

LFS $\stackrel{\mathrm{def}}{=}$ {log-full subgroups of $\pi_1(U_{T_2} \times_k \bar{k})^{(p)}$}.

LFS $\stackrel{\mathrm{def}}{=} \{D \cap \pi_1(U_{T_2} \times_k \bar{k})^{(p)} \subseteq \pi_1(U_{T_2} \times_k \bar{k})^{(p)} \mid D \in \mathcal{D}_{T_2},\ D \cap \pi_1(U_{T_2} \times_k \bar{k})^{(p)} \xrightarrow{\sim} \mathbb{Z}_p^{\oplus 2}\}$

GFS $\stackrel{\mathrm{def}}{=}$ {generalized fiber subgroups of $\pi_1(U_{T_2} \times_k \bar{k})^{(p)}$}

For $F \in$ GFS, $\pi_1(U_{T_2} \times_k \bar{k})^{(p)}/F \xrightarrow{\sim} \pi_1(U_T \times_k \bar{k})^{(p)}$

Proposition

$(0, 3, 2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2}) \rightsquigarrow$ GFS

(pf.) There exist two reconstruction.

- [HMM], Theorem 2.5, (iv),
- [Hgsh3], Theorem A, (v), (using LFS)

18. **From PGCS to CFS-3**

Fix $E \in \mathrm{GFS}$

$\pi_1(U_T \times_k \bar{k})^{(p)} \overset{\text{def}}{=} \pi_1(U_{T_2} \times_k \bar{k})^{(p)}/E$

$\pi_1(U_T)^{(p)} \overset{\text{def}}{=} \pi_1(U_{T_2})^{(p)}/E$

$p^{\pi}_{2/1} \colon \pi_1(U_{T_2})^{(p)} \to \pi_1(U_T)^{(p)}$: the natural quotient homomorphism.

$$T_2(K) \overset{\text{def}}{=} \{\pi_1(U_{T_2} \times_k \bar{k})^{(p)}\text{-conjugacy class of subgroups} \in \mathcal{D}_{T_2}\}$$

where, $k \subseteq K \subseteq \bar{k}$: the maximal pro-$p$ subextension of $\bar{k}/k$

i.e., $G_k^{(p)} = \mathrm{Gal}(K/k)$

$$\mathcal{D}_T \overset{\text{def}}{=} \{C_{\pi_1(U_T)^{(p)}}(p^{\pi}_{2/1}(D)) \mid D \in \mathcal{D}_{T_2}\}$$

where $C_{\pi_1(U_T)^{(p)}}(-)$ denotes the commensurator of $(-)$ in $\pi_1(U_T)^{(p)}$

$$T(K) \overset{\text{def}}{=} \{\pi_1(U_T \times_k \bar{k})^{(p)}\text{-conjugacy class of subgroups} \in \mathcal{D}_T\}$$

$$T(K) \setminus U_T(K) \overset{\text{def}}{=} \{[D] \in T(K) \mid D \cap \pi_1(U_T \times_k \bar{k})^{(p)} \neq \{1\}\} \subseteq T(K)$$

where $T(K) \setminus U_T(K) = \{0, 1, \infty\}$

19. **From PGCS to CFS-4**

$$\mathrm{Aut}_k(U_{T_2}) \subseteq \mathrm{Aut}(T_2(K))$$

for the group of bijections $T_2(K) \xrightarrow{\sim} T_2(K)$ induced by the group $\mathrm{Out}_{G_k^{(p)}}(\pi_1(U_{T_2})^{(p)})$ of $\pi_1(U_{T_2} \times_k \bar{k})^{(p)}$-outer automorphisms of the profinite group $\pi_1(U_{T_2})^{(p)}$ lying over the identity automorphism of $G_k^{(p)}$

$p^{\pi}_{2/1} \colon \pi_1(U_{T_2})^{(p)} \to \pi_1(U_T)^{(p)}$ induces $p^{T}_{2/1} \colon T_2(K) \to T(K)$

$\{T_2(K) \to T(K)\}$ for the $\mathrm{Aut}_k(U_{T_2})$-orbit of $p^{T}_{2/1}$

We construct CFS-collection

$$(T_2(K), T(K), T(K) \setminus U_T(K), \mathrm{Aut}_k(U_{T_2}), \{T_2(K) \to T(K)\})$$

20. **From PGCS to CFS-5**

Key Lemma

$$T_2(K) \xrightarrow{\sim} \{\pi_1(U_{T_2} \times_k \bar{k})^{(p)}\text{-conjugacy class of subgroups} \in \mathcal{D}_{T_2}\}$$

Surj: immediately
Inj: By pro-p version of [Topics], Theorem 4.12 (relative bi)

Key Lemma

$$\mathrm{Aut}_k(U_{T_2}) \xrightarrow{\sim} \mathrm{Out}_{G_k^{(p)}}(\pi_1(U_{T_2})^{(p)})$$

By pro-p version of [Topics], Theorem 4.12 (relative bi)

21. **From PGCS to CFS-6**

Theorem

$(0,3,2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$
$\rightsquigarrow$ CFS-collection $(T_2(K), T(K), T(K) \setminus U_T(K), \mathrm{Aut}_k(U_{T_2}), \{T_2(K) \to T(K)\})$

Not done yet

$\rightsquigarrow$ $0, 1, \infty \in T(K) \setminus U_T(K)$ $\quad$ $(T(K) \setminus U_T(K) = \{0, 1, \infty\})$
$\rightsquigarrow$ $T(K) \setminus \{\infty\}$: additive law and mult. law $(T(K) \setminus \{\infty\} = \mathbb{P}^1(K) \setminus \{\infty\} = K)$

Continue to Part II

22. **Idea-1: Highlights of mono-anabelian geometry**

The highlight is to reconstruct a base field k

There exists a few result

Mochizuki, [AbsTpIII], Corollary 1.10, mono profinite GC

More details: 1 September, 2021

START: $\pi_1(U_X)$

- $\rightsquigarrow$ $\pi_1(U_X) \twoheadrightarrow G_k$ ([AbsTpI], Corollary 2.8)
- $\rightsquigarrow$ the set X (Belyi cuspidalization)
- $\rightsquigarrow$ $k^\times \subseteq \widehat{k^\times} \stackrel{\text{def}}{=} \mathrm{H}^1(G_k, \hat{\mathbb{Z}}(1))$, and $k^\times$: multi. law (Kummer theory)
- $\rightsquigarrow$ $k^\times$, $\mathrm{Div}(X) \subseteq \oplus \mathrm{H}^1$: additive law (Uchida's Lemma)

23. **Idea-2: Past results**

Mochizuki's result: mono profinite GC
I consider the pro-p version
But...

Try to update to the pro-p version
START: $\pi_1(U_X)^{(p)}$
- reconstruct $G_k^{(p)}$: can't

(Note that $\pi_1(U_X)^{(p)} \rightsquigarrow G_k^{(p)}$, by Tsujimura)
- using Belyi cuspidalization: can't
- using Kummer theory: can't

In particular, $\not\rightsquigarrow$ $k^\times$ and mult. law

24. **Idea-3**

Today's theorem

START: $(0,3,2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

$\rightsquigarrow (T_2(K), T(K), T(K) \setminus U_T(K), \mathrm{Aut}_k(U_{T_2}), \{T_2(K) \to T(K)\})$ (by Part I)

- $G_k^{(p)}$: assumption
- $\rightsquigarrow T(K)$ (using $\mathcal{D}_{T_2}$)

where $T(K) = K \sqcup \{\infty\}$ (the curve vs. the base field)

(cf. Mochizuki, [AbsTpII], Corollary 2.9, absolute bi profinite GC equipped with decomposition groups)

25. **Idea-4**

additive law? mult. law?

Using curves

$\mathrm{Aut}_k(U_T) \xrightarrow{\sim} S_3$ (small)

For $a \in U_T$, $a \mapsto a, \frac{1}{a}, 1-a, \frac{1}{1-a}, \frac{a-1}{a}, \frac{a}{a-1}$

Think of higher dimension!

Using second configuration spaces

$\mathrm{Aut}_k(U_{T_2}) \xrightarrow{\sim} S_5$ (big)

For $(a, b) \in U_{T_2}$, $(a, b) \mapsto (\frac{1}{a}, \frac{b}{a}), (\frac{b-a}{b}, \frac{b-a}{b-1}), \ldots$

Construct additive law and mult. law!

(cf. $G_k \not\rightsquigarrow k$, but $\pi_1(U_X) \rightsquigarrow k$ by Mochizuki, [AbsTpIII], Corollary 1.10)

26. **Idea5: Summary**

$$\pi_1(U_T)^{(p)} \overset{\not\rightsquigarrow ?}{\cdots\cdots\cdots} G_k^{(p)}, T(K)$$

$$(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2}) \overset{\rightsquigarrow}{\cdots\cdots} G_k^{(p)}, T(K)$$

Why genus 0 curve? $\Longrightarrow$ the curve vs. the base field ($T(K) = K \sqcup \{\infty\}$)

Why configuration spaces? $\Longrightarrow$ If we use GC for curves, configuration spaces will be appropriate

Why (0,3,2)? (second configuration spaces of $(0,3)$ curves) $\Longrightarrow$

• the set of automorphisms $\mathrm{Aut}_k(U_{T_2})$ ($\xrightarrow{\sim} S_5$) is bigger than others $\mathrm{Aut}_k(U_{X_2})$ ($\xrightarrow{\sim} S_2$)

• $(0,3,2)$ configuration spaces are contained in most configuration spaces (i.e., trip. very ample)

$\Longrightarrow$ START $(0,3,2)$ PGCS $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

27. Relationship between configuration space T_2 and $0, 1, \infty \in T$

$$\begin{array}{ccc} T_2 & \xrightarrow{\sim} & \overline{\mathcal{M}}_{0,5} \\ {\scriptstyle p_2^T}\downarrow & & \downarrow{\scriptstyle p_5} \\ T & \xrightarrow{\sim} & \overline{\mathcal{M}}_{0,4} \end{array} \qquad \begin{array}{ccc} (x,y) & \cdots\cdots & (0,1,\infty,x,y) \\ \vdots & & \vdots \\ x & \cdots\cdots & (0,1,\infty,x) \end{array}$$

Proposition

Let $x \in T(K) \setminus U_T(K) \quad (= \{0, 1, \infty\})$

- $x = 0 \iff x = p_5 \circ (2\ 3)(x, y)$ for $y \in U_T(K)$
- $x = 1 \iff x = p_5 \circ (1\ 3)(x, y)$ for $y \in U_T(K)$
- $x = \infty \iff x = p_5 \circ (1\ 2)(x, y)$ for $y \in U_T(K)$

where, $(2\ 3), (1\ 3), (1\ 2) \in S_5$ and $(x, y) \in T_2(K)$

Relationship between configuration space T_2 and $0, 1, \infty \in T$ is determined by the labeling $S_5 \xrightarrow{\sim} \mathrm{Aut}_k(U_{T_2}) \curvearrowright T_2$ and projections $p_1, \ldots, p_5$

28. **From CFS to base fields-1**

START:

CFS-collection $(T_2(K), T(K), T(K)\setminus U_T(K), \mathrm{Aut}_k(U_{T_2}), \{T_2(K)\to T(K)\})$

For ${}^{\dagger}\lambda, {}^{\ddagger}\lambda \in \{T_2(K)\to T(K)\}$

$${}^{\dagger}\lambda \sim {}^{\ddagger}\lambda \overset{\text{def}}{\Longleftrightarrow} \{{}^{\dagger}\lambda^{-1}(b)\}_{b\in T(K)\setminus U_T(K)} = \{{}^{\ddagger}\lambda^{-1}(b)\}_{b\in T(K)\setminus U_T(K)}$$

CFS $\rightsquigarrow \{T_2(K)\to T(K)\}_{\sim}$

Note that $S_5 \overset{\sim}{\to} \mathrm{Aut}_k(U_{T_2}) \curvearrowright T_2(K)$, $S_3 \overset{\sim}{\to} \mathrm{Aut}_k(U_T) \curvearrowright T(K)$

$\sharp\{T_2(K)\to T(K)\} = 30$, $\{T_2(K)\to T(K)\} = \bigsqcup S_3 \circ p_i$,

$\sharp\{T_2(K)\to T(K)\}_{\sim} = 5$, $\{T_2(K)\to T(K)\}_{\sim} = \{S_3\circ p_i\}$

Let $\phi\colon \mathrm{Aut}_k(U_{T_2}) \overset{\sim}{\to} S_5$: fix any isom.

Write $\{T_2(K)\to T(K)\}_a \in \{T_2(K)\to T(K)\}_{\sim}$ for the unique class s.t.

$$\{T_2(K)\to T(K)\}_a = \{T_2(K)\to T(K)\}_a \circ (\phi^{-1}(b\ c))$$

for all transpositions $(b\ c)\in S_5$ s.t. $a\notin\{b,c\}$

If $\phi\colon \mathrm{Aut}_k(U_{T_2}) \overset{\sim}{\to} S_5$: canonical, then $\{T_2(K)\to T(K)\}_i = S_3\circ p_i \ni p_i$

29. **From CFS to base fields-2**

Let $\lambda \in \{T_2(K) \to T(K)\}_1$: fix

$p_2 \overset{\text{def}}{=} \lambda \circ (\phi^{-1}(1\ 2))$, $p_i \overset{\text{def}}{=} p_{i-1} \circ (\phi^{-1}(i-1\ i))$

Let $x, y \in T(K)$ be distinct elements. Write $(x,y) \in T_2(K)$ for the unique element such that $p_5(x,y) = x$, $p_4(x,y) = y$

Write 0 for the unique element $x \in T(K) \setminus U_T(K)$ s.t. $x = p_5 \circ (\phi^{-1}(2\ 3))(x,y)$ for every $y \in T(K) \setminus \{x\}$

Write 1 for the unique element $x \in T(K) \setminus U_T(K)$ s.t. $x = p_5 \circ (\phi^{-1}(1\ 3))(x,y)$ for every $y \in T(K) \setminus \{x\}$

Write ∞ for the unique element $x \in T(K) \setminus U_T(K)$ s.t. $x = p_5 \circ (\phi^{-1}(1\ 2))(x,y)$ for every $y \in T(K) \setminus \{x\}$

Relationship between configuration space T_2 and $0, 1, \infty \in T$ is determined by the labeling $S_5 \xrightarrow{\sim} \mathrm{Aut}_k(U_{T_2}) \curvearrowright T_2$ and projections $p_1, \ldots, p_5$ (cf. p.27)

30. **From CFS to base fields-3**

- We write

$$\tau_{\mathrm{rf}} \overset{\mathrm{def}}{=} \phi^{-1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix}$$

- We write

$$\tau_{\mathrm{ra}} \overset{\mathrm{def}}{=} \phi^{-1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$

- We write

$$\tau_{\mathrm{cr}} \overset{\mathrm{def}}{=} \phi^{-1}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$$

Note that if $\phi\colon \mathrm{Aut}_k(U_{T_2}) \xrightarrow{\sim} S_5$: canonical then

$$\tau_{\mathrm{rf}}\colon (x,y) \mapsto (0,1,\infty,x,y) \underset{\tau_{\mathrm{rf}}}{\mapsto} (0,1,x,\infty,y) \underset{\frac{t(1-x)}{t-x}}{\mapsto} \left(0,1,\infty,1-x,\frac{y(x-1)}{x-y}\right) \mapsto \left(1-x,\frac{y(x-1)}{x-y}\right)$$

$$\tau_{\mathrm{ra}}\colon (x,y) \mapsto (0,1,\infty,x,y) \underset{\tau_{\mathrm{ra}}}{\mapsto} (0,x,\infty,1,y) \underset{\frac{t}{x}}{\mapsto} \left(0,1,\infty,\frac{1}{x},\frac{y}{x}\right) \mapsto \left(\frac{1}{x},\frac{y}{x}\right)$$

$$\tau_{\mathrm{cr}}\colon (x,y) \mapsto (0,1,\infty,x,y) \underset{\tau_{\mathrm{cr}}}{\mapsto} (\infty,x,y,0,1) \underset{\frac{y-x}{y-t}}{\mapsto} \left(0,1,\infty,\frac{y-x}{y},\frac{y-x}{y-1}\right) \mapsto \left(\frac{y-x}{y},\frac{y-x}{y-1}\right)$$

31. From CFS to base fields-4

We shall say that a collection of maps

$$+,\times\colon (T(K)\setminus\{\infty\})\times(T(K)\setminus\{\infty\})\to(T(K)\setminus\{\infty\})$$
$$-\colon (T(K)\setminus\{\infty\})\to(T(K)\setminus\{\infty\})$$
$$/\colon (T(K)\setminus\{0,\infty\})\to(T(K)\setminus\{0,\infty\})$$

is CFS-admissible if the following conditions are satisfied:

$$-0=0,\quad /1=1$$
$$\text{For } x,y\in T(K)\setminus\{\infty\},\ x+y=y+x,\ x\times y=y\times x$$
$$\text{For } x\in T(K)\setminus\{\infty\},\ 0+x=x,\ 0\times x=0,\ 1\times x=x,\ x+(-x)=0$$
$$\text{For } x\in T(K)\setminus\{0,\infty\},\ x\times(/x)=1$$
$$\text{Let } x,y\in U_T(K) \text{ s.t. } x\neq y.\ \text{ Then } /x=p_5(\tau_{\mathrm{ra}}(x,y))$$
$$\text{Let } x,y\in U_T(K) \text{ s.t. } y\neq /x.\ \text{ Then } x\times y=p_4(\tau_{\mathrm{ra}}(/x,y))$$

Note that

$$(x,y)\underset{\tau_{\mathrm{ra}}}{\mapsto}\left(\frac{1}{x},\frac{y}{x}\right)\underset{p_5}{\mapsto}\frac{1}{x}$$
$$(\tfrac{1}{x},y)\underset{\tau_{\mathrm{ra}}}{\mapsto}(x,xy)\underset{p_4}{\mapsto}xy$$

$$\text{There exists an element } x\in U_T(K) \text{ s.t. } /x=x$$
$$\text{Let } x,y\in U_T(K) \text{ s.t. } /x=x.\ \text{ Then } -1=x\in T(K),\ -y=x\times y$$
$$\text{Let } x\in U_T(K)\setminus\{-1\}.\text{ Then } 1+1=p_5(\tau_{\mathrm{rf}}(-1,x))$$
$$\text{Let } x\in U_T(K)\setminus\{-1\}.\text{ Then } x+1=p_5(\tau_{\mathrm{cr}}(x,-1))$$
$$\text{Let } x\in T(K)\setminus\{\infty\} \text{ s.t. } y\neq 0.\text{ Then } x+y=y\times(\tfrac{x}{y}+1)$$

Note that

$$(-1,x)\underset{\tau_{\mathrm{rf}}}{\mapsto}\left(1-(-1),\frac{x(-1-1)}{-1-x}\right)\underset{p_5}{\mapsto}1-(-1)=2$$
$$(x,-1)\underset{\tau_{\mathrm{cr}}}{\mapsto}\left(\frac{-1-x}{-1},\frac{-1-x}{-1-1}\right)\underset{p_5}{\mapsto}\frac{-1-x}{-1}=x+1$$

32. **From CFS to base fields-5**

If one fixes the data $(\mathrm{CFS}, \phi, \lambda)$, then any CFS-admissible collection of maps is unique

Thus, if the data $(\mathrm{CFS}, \phi, \lambda)$ admits a CFS-admissible collection of maps, then we shall write

$$K = (T(K) \setminus \{\infty\}, +, \times, -, /)$$

for the set $T(K) \setminus \{\infty\}$ equipped with the maps $+, \times, -, /$

33. **From CFS to base fields-6: Main Theorems**

START: $(0,3,2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

γ: $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2}) \xrightarrow{\sim} (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$: any isom.

By Part I, PGCS $\rightsquigarrow$ CFS

In particular, γ induces

$(T_2(K), T(K), T(K) \setminus U_T(K), \mathrm{Aut}_k(U_{T_2}), \{T_2(K) \to T(K)\})$

$\xrightarrow{\sim} (T_2(K), T(K), T(K) \setminus U_T(K), \mathrm{Aut}_k(U_{T_2}), \{T_2(K) \to T(K)\})$

<u>Theorem 3.13, (i)</u>

If ϕ: $\mathrm{Aut}_k(U_{T_2}) \xrightarrow{\sim} S_5$, $\lambda \in \{T_2(K) \to T(K)\}_1$: good choices

i.e.,

$$\begin{array}{ccc} \mathrm{Aut}_k(U_{T_2}) & \xrightarrow{\sim} & \mathrm{Aut}_k(U_{T_2}) \\ & {\scriptstyle \phi}\searrow^{\sim} & \downarrow {\scriptstyle \text{canonical}} \\ & & S_5 \end{array}$$

$$\{T_2(K) \to T(K)\}_1 \xrightarrow{\sim} \{T_2(K) \to T(K)\}_1 = S_3 \circ p_1$$
$$\lambda \mapsto p_1$$

Then

$$T(K) \xrightarrow{\sim} T(K) = K \sqcup \{\infty\}$$

induces

$$0,\ 1,\ \infty \mapsto 0, 1, \infty,$$

and the field structure of K determines a CFS-admissible collection of maps.

In particular, K may be regarded as a field

34. **From CFS to base fields-7**

Definition

Let $\square \in \{\dagger, \dagger\}$

$^{\square}\mathscr{A} = (^{\square}T_2(K), {}^{\square}T(K), {}^{\square}(T(K) \setminus U_T(K)), {}^{\square}\mathrm{Aut}_k(U_{T_2}), {}^{\square}\{T_2(K) \to T(K)\})$: CFS-collections

$(\alpha, \beta) \colon {}^{\dagger}\mathscr{A} \xrightarrow{\sim} {}^{\ddagger}\mathscr{A}$: an isom. of CFS

$\overset{\text{def}}{\iff} \alpha \colon {}^{\dagger}T_2(K) \xrightarrow{\sim} {}^{\ddagger}T_2(K)$, $\beta \colon {}^{\dagger}T(K) \xrightarrow{\sim} {}^{\ddagger}T(K)$: bijections of sets s.t.

$\beta({}^{\dagger}(T(K) \setminus U_T(K))) = {}^{\ddagger}(T(K) \setminus U_T(K))$

$\alpha \circ {}^{\dagger}\mathrm{Aut}_k(U_{T_2}) \circ \alpha^{-1} = {}^{\ddagger}\mathrm{Aut}_k(U_{T_2})$

$\beta \circ {}^{\dagger}\{T_2(K) \to T(K)\} \circ \alpha^{-1} = {}^{\ddagger}\{T_2(K) \to T(K)\}$

Proposition

Let $^{\square}\phi$, $^{\square}\Lambda$. Then there exists an isom. of CFS (α, β) s.t. $(\alpha, \beta)^{\dagger}\phi = {}^{\ddagger}\phi$, $(\alpha, \beta)^{\dagger}\lambda = {}^{\ddagger}\lambda$

35. **From CFS to base fields-8**

Theorem 3.13, (ii)

Let ${}^{\square}\mathscr{A}, {}^{\square}\phi, {}^{\square}\lambda, (\alpha, \beta)$

Suppose that $(\alpha, \beta)^{\dagger}\phi = {}^{\ddagger}\phi,\ (\alpha, \beta)^{\dagger}\lambda = {}^{\ddagger}\lambda$

and $({}^{\dagger}\mathscr{A}, {}^{\dagger}\phi, {}^{\dagger}\lambda)$ admits a CFS-admissible collection of maps

Then $({}^{\ddagger}\mathscr{A}, {}^{\ddagger}\phi, {}^{\ddagger}\lambda)$ admits a CFS-admissible collection of maps

Moreover, β induces a field isom. ${}^{\dagger}K \xrightarrow{\sim} {}^{\ddagger}K$ (cf. (i))

36. **From CFS to base fields-9**

Definition

$\mathrm{Aut}_k(U_T) \stackrel{\mathrm{def}}{=} \{\sigma \in \mathrm{Aut}(U_T(K)) \mid \exists \tau \in S_5 \text{ s.t. } \tau(1) = 1,\ \sigma \circ \lambda = \lambda \circ \phi^{-1}(\tau)\}$

$$\mathrm{Aut}_k(U_T) \xrightarrow{\sim} \mathrm{Aut}(T(K) \setminus U_T(K)) \xrightarrow{\sim} S_3$$

$$\sigma \mapsto \sigma|_{T(K)\setminus U_T(K)}$$

Theorem 3.13, (iii) Let $\mathscr{A}, {}^{\square}\phi, {}^{\square}\lambda$

Then there exists a unique element $\sigma \in \mathrm{Aut}_k(U_T)$ s.t.

$$\sigma({}^{\dagger}0, {}^{\dagger}1, {}^{\dagger}\infty) = ({}^{\ddagger}0, {}^{\ddagger}1, {}^{\ddagger}\infty)$$

and σ determines a field isom.

$${}^{\dagger}K \xrightarrow{\sim} {}^{\ddagger}K$$

(cf. (ii))

37. **From CFS to base fields-10**

Theorem 4.8, (iii)

Let $(0,3,2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$
$\gamma\colon (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2}) \xrightarrow{\sim} (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$: any isom. of PGCS
Then γ induces an isom. of CFS
$(\alpha, \beta)\colon (T_2(K), T(K), T(K) \setminus U_T(K), \mathrm{Aut}_k(U_{T_2}), \{T_2(K) \to T(K)\})$
$\xrightarrow{\sim} (T_2(K), T(K), T(K) \setminus U_T(K), \mathrm{Aut}_k(U_{T_2}), \{T_2(K) \to T(K)\})$
Write

$$t_{\beta(0),\beta(1),\beta(\infty)} \in \Gamma(U_T, \mathcal{O})$$

for the regular function s.t.

- $t_{\beta(0),\beta(1),\beta(\infty)}$ induces a bijection
$$t_{\beta(0),\beta(1),\beta(\infty)}\colon T(K) \xrightarrow{\sim} K \sqcup \{\infty\}$$
- the zero divisor is of degree 1 and supported on $\beta(0)$
- $t_{\beta(0),\beta(1),\beta(\infty)}(\beta(1)) = 1$
- the divisor of poles is of degree 1 and supported on $\beta(\infty)$

(i.e., a functional linear transformation $x \mapsto \frac{x-\beta(0)}{x-\beta(\infty)} \frac{\beta(1)-\beta(\infty)}{\beta(1)-\beta(0)}$)

Then the bijection

$$T(K) \xrightarrow{\sim}_{\beta} T(K) \xrightarrow{\sim}_{t_{\beta(0),\beta(1),\beta(\infty)}} K \sqcup \{\infty\}$$

determines a field isom.

$$K \xrightarrow{\sim} K$$

that is equivariant with respect to the respective natural actions of the profinite groups $G_k^{(p)}, G_k^{(p)}$, relative to the isom. $\gamma|_G\colon G_k^{(p)} \xrightarrow{\sim} G_k^{(p)}$

38. **From CFS to base fields-10: Summary**

Theorem 3.13, (i)

If ϕ, λ: good, then $(\mathscr{A}, \phi, \lambda)$ admits CFS-adm. maps

Theorem3.13, (ii)

Any $(\mathscr{A}, \phi, \lambda)$ admits CFS-adm. maps

Theorem3.13, (ii)

$0, 1, \infty$ determine the field structure of K

$\exists \sigma \in \mathrm{Aut}_k(U_T)$ s.t.

$$\text{the field of } (\mathscr{A}, {}^{\dagger}\phi, {}^{\dagger}\lambda) \xrightarrow{\sim}_{\sigma} \text{the field of } (\mathscr{A}, {}^{\dagger}\phi, {}^{\dagger}\lambda)$$

Theorem 4.8, (iii), (Theorem 0.3)

PGCS $\mathscr{B} \rightsquigarrow K \curvearrowleft G_k^{(p)}$

Write $k \stackrel{\text{def}}{=} K^{G_k^{(p)}}$, then PGCS $\mathscr{B} \rightsquigarrow k$

39. Reconstruction of $(0, 3, 2)$ PGCS-1

Definition

We shall say that $(X^{\log}, n)$ is tripodally very ample if one of the following conditions (i), (ii), (iii) holds:

(i) $\sharp(X(k) \setminus U_X(k)) = 3$ and $(g, r, n) = (0, 3, 2)$

(ii) $X(k) \setminus U_X(k) \neq \emptyset$, $n > 2$, and $r > 0$

(iii) $U_X(k) \neq \emptyset$ and $n > 3$

Note

(i) ok

(ii) We consider log divisors

We shall refer to an irreducible divisor V of X_n contained in the complement $X_n \setminus U_{X_n}$ as a log divisor of $X_n^{\log}$

Since $r > 0$, $V^{\log\leq 1}$ is isomorphic to $U_{T_m} \times_k U_{X_{n-1-m}}$ (cf. [Hgsh1], Lemma 6.1)

If $n = 3$, then $\exists V$: a log divisor s.t. $V^{\log\leq 1}$ is isomorphic to U_{T_2}

$\Longrightarrow$ It seems possible?

(iii) Suppose that $r = 0$

If $n = 4$, then $\exists V$: a log divisor s.t. $V^{\log\leq 1}$ is isomorphic to $U_{T_2} \times U_X$

$\Longrightarrow$ It seems possible??

trip. very ample $\Longleftrightarrow$ "$T_2^{\log} \subseteq X_n^{\log}$"

40. **Reconstruction of** $(0,3,2)$ **PGCS-2: More detail**

(ii) $c \in X(k) \setminus U_X(k)$: assumption

$$X^{\log} \longleftarrow X_2^{\log} \longleftarrow X_3^{\log} \longleftarrow X_n^{\log}$$

$$\begin{array}{ccccc} c & \cdots\cdots & U_T & \cdots\cdots & U_{T_2} \\ & \ddots & \sqcup & \ddots & \sqcup \\ & & U_X & & U_T \times_k U_X \end{array}$$

(iii) $c \in U_X(k)$: assumption

$$X_2^{\log} \longleftarrow X_3^{\log} \longleftarrow X_4^{\log} \longleftarrow X_n^{\log}$$

$$\begin{array}{ccccc} (c,c) & \cdots\cdots & U_T & \cdots\cdots & U_{T_2} \\ & \ddots & \sqcup & \ddots & \sqcup \\ & & U_X & & U_T \times_k U_X \end{array}$$

$\Longrightarrow$ It seems possible!

41. **Reconstruction of** $(0, 3, 2)$ **PGCS-3**

(g, r, n) PGCS-collection $(\pi_1(U_{X_n})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_n})$

$\rightsquigarrow$ LFS (LFS $\stackrel{\text{def}}{=}$ {log-full subgroups}) (cf. p.17)

$\rightsquigarrow$ LD (LD $\stackrel{\text{def}}{=}$ {inertia subgroups ass'd to log divisors}) (cf. [Hgsh3], Theorem A, (i))

So we can consider log divisors

(ii)

• How do we choose U_{T_2} from $\{U_{T_2}, U_T \times_k U_X\}$?

Since $\pi_1(U_{T_2} \times_k \bar{k})^{(p)}$ is indecomposable, and $\pi_1(U_T \times_k \bar{k})^{(p)} \times \pi_1(U_X \times_k \bar{k})^{(p)}$ is decomposable (cf. [Hgsh1], §6; [Ind], Theorem 3.5)

So we choose U_{T_2}

42. **Reconstruction of** $(0,3,2)$ **PGCS-4**

(ii)

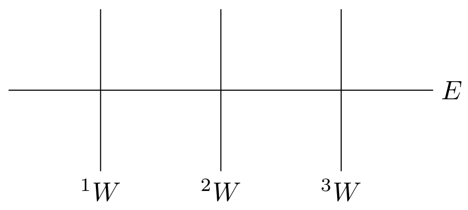
• How do we choose U_T from $\{U_T, U_X\}$? Since $\pi_1(U_T \times_k \bar{k})^{(p)}$ is isom. to $\pi_1(U_X \times_k \bar{k})^{(p)} \iff (g,r) = (0,3),(1,1)$

(Note that U_X is hyperbolic curve of type (g,r))

If the case $(g,r) = (0,3)$, there's nothing to do.

So we consider $(g,r) = (1,1)$ and $(g,r,n) = (1,1,2)$

Since $(g,r,n) = (1,1,2)$, then $\sharp\mathrm{LD} = 4$



$^{\square}W^{\log\leq 1}$ is isom. to U_X

$E^{\log\leq 1}$ is isom. to U_T

So we choose E and U_T

43. **Reconstruction of** $(0, 3, 2)$ **PGCS-6: Summary**

START: trip. very ample (g, r, n) PGCS-collection $(\pi_1(U_{X_n})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_n})$
$\rightsquigarrow$ GFS (cf. p.17)
For $F \in$ GFS,

$$1 \to F \to \pi_1(U_{X_n})^{(p)} \to \pi_1(U_{X_m})^{(p)} \to 1$$

Thus, $\rightsquigarrow$ trip. very ample (g, r, m) PGCS-collection $(\pi_1(U_{X_m})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_m})$
$\rightsquigarrow$ $(0, 3, 2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$
(cf. [Hgsh2], Remark 6.3; [NodNon], Theorem A)

44. **Recent research-1**

trip. very ample $\Longleftrightarrow$ "$T_2^{\log} \subseteq X_n^{\log}$"

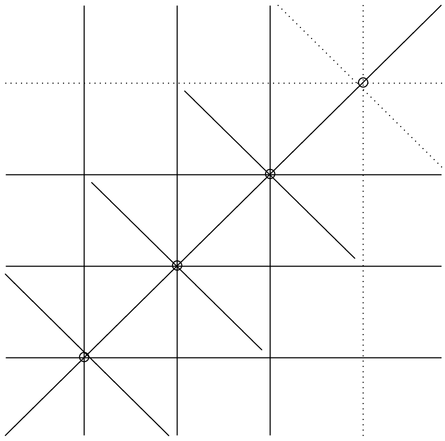
In recent research, the reconstruction can be applied in situation "$T_2^{\log} \supseteq X_n^{\log}$", e.g.,

$(0, r, 2)$ PGCS-collection $(\pi_1(U_{X_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_2})$ $\quad (r \geq 3)$

$\rightsquigarrow$ $(0, 3, 2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

Since the difference between $T_2^{\log}$ and $X_2^{\log}$ is a few log divisors (cf. [Hgsh3], §2)

45. **Recent research-2**



the number of log divosrs of $(g, r, n) = (0, 3, 2)$ is 10 (all lines)
the number of log divosrs of $(0, 4, 2)$ is 13 (all lines and all dotted lines)
The difference is only three dotted lines, so it seems possible
$(0, 4, 2)$ PGCS-collection $(\pi_1(U_{X_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{X_2})$
$\rightsquigarrow (0, 3, 2)$ PGCS-collection $(\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$

46. **Semi-absolute bi pro-p GC-1**

Theorem 0.1
$^{\square}k$: generalized sub-p-adic field
$^{\square}\mathscr{B} \overset{\mathrm{def}}{=} ({}^{\square}\pi_1(U_{X_n})^{(p)}, {}^{\square}G_k^{(p)}, {}^{\square}\mathcal{D}_{X_n})$: trip. very ample $({}^{\square}g, {}^{\square}r, {}^{\square}n)$ PGCS-collection
Then the natural morphism

$$\mathrm{Isom}({}^{\dagger}U_{X_n}, {}^{\ddagger}U_{X_n}) \to \mathrm{Isom}({}^{\dagger}\mathscr{B}, {}^{\ddagger}\mathscr{B})/\mathrm{Inn}$$

is bijective

proof. First, we consider the following Claim

Claim
PGCS-collection $\mathscr{B} \rightsquigarrow (g, r, n)$
(pf.) There exist two reconstruction.
- [HMM], Theorem A, (i),
- [Hgsh3], Theorem A, (ii), (using LFS)

by Claim, write $(g, r, n) \overset{\mathrm{def}}{=} ({}^{\square}g, {}^{\square}r, {}^{\square}n)$

47. **Semi-absolute bi pro-p GC-2**

Since $^{\square}G_k^p$ is center-free (cf. [Hgsh2], Lemma 5.3; [LocAn], Lemma 15.8)

$$\mathrm{Isom}({}^{\dagger}U_{X_n}, {}^{\ddagger}U_{X_n}) \to \mathrm{Isom}({}^{\dagger}\mathscr{B}, {}^{\ddagger}\mathscr{B})/\mathrm{Inn}$$

is injective

Using

trip. very ample(g, r, n) PGCS-collection $\mathscr{B} \rightsquigarrow (0, 3, 2)$ PGCS-collection $\rightsquigarrow k$

and pro-p version of [Topics], Theorem 4.12 (relative bi)

$$\mathrm{Isom}({}^{\dagger}U_{X_n}, {}^{\ddagger}U_{X_n}) \to \mathrm{Isom}({}^{\dagger}\mathscr{B}, {}^{\ddagger}\mathscr{B})/\mathrm{Inn}$$

is bijective

48. **Reconstruct function fields-1**

START: $(0,3,2)$ PGCS-collection $\mathscr{B} = (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$ and ${}^{\dagger}\pi_1(U_{T_2})$

By Mochizuki's mono profinite GC

${}^{\dagger}\pi_1(U_{T_2}) \rightsquigarrow {}^{\dagger}\pi_1(U_T),\ {}^{\dagger}G_k,\ \mathrm{Fnct}(Z),\ {}^{\dagger}\bar{k},\ {}^{\dagger}\mathscr{B} = ({}^{\dagger}\pi_1(U_{T_2})^{(p)}, {}^{\dagger}G_k^{(p)}, {}^{\dagger}\mathcal{D}_{T_2})$

where Z denotes the profinite étale covering of U_T s.t. $\pi_1(U_T) = \mathrm{Gal}(Z/U_T)$

Let $\delta\colon \mathscr{B} \xrightarrow{\sim} {}^{\dagger}\mathscr{B}$ isom. of PGCS and $E \in \mathrm{GFS}$ (cf. p.17)

$$\begin{array}{ccccccc}
\pi_1(U_{T_2})^{(p)} & & \longrightarrow\!\!\!\!\rightarrow & & \pi_1(U_T)^{(p)} & \longrightarrow\!\!\!\!\rightarrow & G_k^{(p)} \\
\delta \downarrow \simeq & & & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\
{}^{\dagger}\pi_1(U_{T_2})^{(p)} & \leftarrow\!\!\!\!\leftarrow\!\!\!\!- & {}^{\dagger}\pi_1(U_{T_2}) & \longrightarrow\!\!\!\!\rightarrow & {}^{\dagger}\pi_1(U_T) & \longrightarrow\!\!\!\!\rightarrow & {}^{\dagger}G_k
\end{array}$$

$$\mathrm{Fnct}(W) \overset{\mathrm{def}}{=} \mathrm{Fnct}(Z)^{\mathrm{Ker}({}^{\dagger}\pi_1(U_T) \twoheadrightarrow \pi_1(U_T)^{(p)})}$$

$$ {}^{\dagger}K \overset{\mathrm{def}}{=} {}^{\dagger}\bar{k}^{\mathrm{Ker}({}^{\dagger}G_k \twoheadrightarrow G_k^{(p)})}$$

where W denotes the profinite étale covering of U_T s.t. $\pi_1(U_T)^{(p)} = \mathrm{Gal}(W/U_T)$

49. **Reconstruct function fields-2**

$$\begin{array}{ccccc} & & & \overset{\text{rec.}}{\nearrow} & K \\ \text{PGCS } \mathscr{B} & & \xrightarrow{\text{rec.}} & & {}^{\dagger}K \subseteq \mathrm{Fnct}(W) \\ \delta \big\downarrow \text{ any isom.} & & & & \cap \\ \text{PGCS } {}^{\dagger}\mathscr{B} & \xleftarrow{\text{rec.}} & {}^{\dagger}\pi_1(U_{T_2}) & \xrightarrow{\text{rec.}} & \mathrm{Fnct}(Z) \end{array}$$

Next, we consider K v.s. ${}^{\dagger}K \subseteq \mathrm{Fnct}(W)$

50. **Reconstruct function fields-3**

Let $T \in \mathrm{Fnct}(W)$, then T induces a map

$$T\colon \mathcal{D}_T \to {}^{\dagger}K \sqcup \{\infty\}$$

There exists a unique element $T \in \mathrm{Fnct}(U_X) \overset{\mathrm{def}}{=} \mathrm{Fnct}(W)^{\pi_1(U_X)^{(p)}}$ s.t.

- the zero divisor of T is of degree 1 and supported on 0
- $T(1) = {}^{\dagger}1$
- the divisor of poles of T is of degree 1 and supported on ∞

then T induces a field isom.

$$K \overset{\sim}{\to} {}^{\dagger}K$$

(Theorem 0.4)

51. **Reconstruct function fields-4**

START: $(0,3,2)$ PGCS-collection $\mathscr{B} = (\pi_1(U_{T_2})^{(p)}, G_k^{(p)}, \mathcal{D}_{T_2})$ and ${}^{\dagger}\pi_1(U_{T_2})$
Let $\delta \colon \mathscr{B} \xrightarrow{\sim} {}^{\dagger}\mathscr{B}$: isom. of PGCS
$E \in \mathrm{GFS}$
Then $\rightsquigarrow$ $K \xrightarrow{\sim} {}^{\dagger}K \subseteq \mathrm{Fnct}(W)$ (Theorem 0.4)
Note that $\sharp\mathrm{GFS} = \sharp\{\text{generalized fiber subgroups}\} = 5$

$$E_{\cap} \stackrel{\text{def}}{=} \bigcap_{E \in \mathrm{GFS}} E, \quad \pi_1(U_{T_2})^{(p)}_{2\to 1} \stackrel{\text{def}}{=} \pi_1(U_{T_2})^{(p)}/E_{\cap}$$

$$\mathrm{Fnct}(W) \hookrightarrow \mathrm{RatMaps}(\mathcal{D}_T, K) \colon T \mapsto T(-)$$

$$\mathrm{Fnct}(W) \hookrightarrow \mathrm{RatMaps}(\mathcal{D}_{T_2}, K)$$

$$T \mapsto T(-) \circ p_{2/1}$$

The field structure of K induces a natural ring structure on $\mathrm{RatMaps}(\mathcal{D}_{T_2}, K)$

52. **Reconstruct function fields-5**

We write R for the subring of $\mathrm{RatMaps}(\mathcal{D}_{T_2}, K)$ generated by the subrings

$$\mathrm{Fnct}(W) \hookrightarrow \mathrm{RatMaps}(\mathcal{D}_{T_2}, K)$$

for $E \in \mathrm{GFS}$

Then R is an integral domain on which the subgroup $E_\cap \subseteq \pi_1(U_{T_2})^{(p)}$ acts trivially

The quotient field $\mathrm{Fnct}(Y) \stackrel{\mathrm{def}}{=} \mathrm{Frac}(R)$ which is equipped with an action by $\pi_1(U_{T_2})^{(p)}_{2\to 1}$

where $Y \to U_{T_2}$ for the profinite étale covering corresponding to $\pi_1(U_{T_2})^{(p)}_{2\to 1}$ (Theorem 0.2)

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